Topology Vol. 28, No. 2, pp. 247–271, 1989
Printed in Great Britain.

0040-9383/89 $03.00+.00

FREELY REDUCING GROUP READINGS FOR 2-COMPLEXES IN 4-MANIFOLDS

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(*Received in revised form* 21 *December* 1987)

§1. INTRODUCTION

WITH ANY polyhedral 2-complex K, there is an associated group presentation, $\mathscr{P}_K = \{x_1, \ldots, x_n | r_1, \ldots, r_p\}$ where the generators $x_1, \ldots, x_n$, correspond to the 1-cells of K not in a maximal tree and the relators r_i are words on the alphabet $\{x_i^{\pm 1}\}$ (not necessarily reduced) that describe the attachment of the 2-cells to the 1-cells. This paper concerns polyhedral deformations of 2-complexes K and the effects of these deformations on the associated group presentations $\mathscr{P}_K$. By a *formal deformation* of a polyhedron K to a polyhedron L, $K \nearrow\!\searrow L$, we mean a sequence of polyhedra $K = K(0), K(1), \ldots, K(m) = L$, where each $K(i+1)$ results from $K(i)$ either by the collapse of a piecewise linear cell of some dimension across a free face or by expansion along some piecewise linear cell. If in the sequence of polyhedra, each $K(i)$ has dimension at most n, then we say that K *n-deforms to* L, $K \nearrow\!\searrow^n L$. The polyhedra are assumed to be abstract polyhedra, they do not have to sit in some Euclidean space or in some manifold. If for some polyhedron X, usually a manifold, it happens that each $K(i) \subset X$, then we say that K *deforms to* L *in* X, $K \nearrow\!\searrow_X L$. If M is a manifold, and if K and L are in the interior of M then, by regular neighborhood theory, $K \nearrow\!\searrow_M L$ means that K and L have isotopically embedded regular neighborhoods. Formal 3-deformations between 2-complexes K and L correspond to extended Nielsen transformation between the associated group presentations $\mathscr{P}_K$ and $\mathscr{P}_L$ (see Wright [25], also [2], [3], [8], [9], [17], [18], [20], [21]).

Perhaps the most simple of the extended Nielsen transformations on the associated group presentations is the deletion of a cancelling pair of syllables in one of the relator words. But if the 2-complex K corresponding to the group presentation is contained in a 4-manifold M, and if we desire a deformation, $K \nearrow\!\searrow_M L$, in M to effect the free reduction, then there are obstacles to finding such a deformation.

Question 1. When does a polyhedral 2-complex K in a 4-manifold M 3-deform in M to a new 2-complex L so that the associated presentation $\mathscr{P}_L$ is obtained from $\mathscr{P}_K$ by freely reducing all the relator words?

We will restrict our attention to the following modified form for Question 1:

Question 2. Same as Question 1 except that the 3-deformation is now required to leave the 1-skeleton of K fixed.

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†Research supported in part by the National Science Foundation and by a grant from the University of Illinois Research Board.

When one deals with abstract formal deformations, finding a deformation that keeps the 1-skeleton of K fixed is not a problem. The next thing to say is that the answer to Question 1 is "not always". Otherwise the Mazur manifold with the dunce hat spine [26] would have to be a 4-ball which it is not. A special case of Question 2 is quite amenable to an analysis by linking obstructions and yet sheds light on the more general Question 2 including the Mazur manifold example:

Question 3. For which 2-complexes K in a 3-manifold M, do the corresponding 2-complexes $K \times 0$ in $M \times [0, 1]$ 3-deform in $M \times [0, 1]$, keeping the 1-skeletons fixed, to 2-complexes L in $M \times [0, 1]$ (though not necessarily in $M \times 0$) so that the presentations $\mathscr{P}_L$ come from the presentations $\mathscr{P}_K$ by freely reducing the relator words.

The answer to Question 3 is still "not all", for Jim Howie has pointed out an example to us where an obstruction to a free reduction problem of the type in Question 3 is necessary in order that the Mazur manifold not be a ball. A whole family of examples of this type will be described at the end of the paper.

Here is an outline of the paper: In §3 we describe a 13 step free reduction construction attempting to answer "for all" to Question 3. In Step III an obstruction, called the *linked cancellation segments* obstruction, will be revealed. It will be the single obstacle preventing the completion of the construction. After we complete the construction under the assumption that the linked cancellation segment obstruction vanishes, we will describe in §4 a twofold application covering the case of many 2-complexes that do not fit the special form of the 3-manifold complexes in Question 3. We close in §5 with some questions and conjectures.

§2. NOTATION AND CONVENTIONS

The notation $N(A, B)$ stands for the closed simplicial neighborhood of A in B where A is a simplex or a subcomplex of the complex B. The neighborhood $N(A, B)$ is the union of all closed simplexes of B that touch A.

Let $K \subset M^3$ be given as in the introduction. Let G denote the 1-skeleton of K, and let the 2-cells be $e_1, \ldots, e_i, \ldots$. Furthermore let J be a regular neighborhood of G that is small relative to the 2-cells e_i (i.e. a second derived neighborhood for a triangulation in which the closure of each e_i is a subcomplex). Set $S = K \cap BdJ$ and for each e_i set $S_i = e_i \cap BdJ$. We may suppose that J carries the structure of a pwl mapping cylinder C_f (see [1] and [6]) where f is a pwl map $f\colon BdJ \to G$ with $K \cap J = C_{f|S}$, and we may suppose that f is locally non-degenerate on S so that the preimage of any point under $f|S$ is finite. This last condition can be achieved by a slight adjustment of f. The pwl mapping cylinder approach of [1] allows us to fix a pwl structure on C_f and still be able to adjust the simplicial structure. To simplify notation suppose that $f\colon BdJ \to G$ is a simplicial map, that S is a subcomplex of BdJ, and that f carries each 1-simplex of S onto a 1-simplex of G. Finally, we may assume that for each 1-simplex σ of G, the preimage $f^{-1}(b(\sigma))$ is a simple closed curve where b denotes barycenter. There are mapping cylinder coordinates for J associated with C_f: $\lambda\colon J \to [0, 1]$($\lambda^{-1}(0) = G$ and $\lambda^{-1}(1) = BdJ$) and $r\colon J \to G$ ($r|BdJ = f$) where λ and r are simplicial maps and $[0, 1]$ has vertices 0 and 1. Thus the mapping cylinder coordinate pair (x, t) describes the point p on the line from x to $f(x)$ with $\lambda(p) = t$ and $r(p) = f(x)$.

CONVENTION. Any time we have an object, say Z, with a mapping cylinder structure and we also have a number $0 < \varepsilon \leq 1$, we will use ε as a superscript, as in Z^ε, to indicate the part of Z whose λ values are in the range $[0, \varepsilon]$. Thus $J^\varepsilon = \lambda^{-1}([0, \varepsilon])$.

We may make the reading $\mathscr{P}_K$ directly from BdJ. Certain 1-simplexes of G are ignored (those in a maximal tree) and the remaining ones are oriented, say $\sigma_1, \ldots, \sigma_n$. The simple closed curves S_j are oriented and given basepoints $*_j$, and then relators r_j are read from the intersections $C_i \cap S_j$ where $C_i = f^{-1}(b(\sigma_i))$. We proceed around S_j in the direction of the orientation and list x_i^+ when S_j passes through C_i from the negative to the positive side and we list x_i^- in the other case. For convenience, when we later pass to $M \times [0, 1]$ we will identify G with $G \times 0$ and K with $K \times 0$.

§3. THE FREE REDUCTION CONSTRUCTION†

I. There are simplicial arcs, $A_1, \ldots, A_k, \ldots$ in S with disjoint interiors not containing any $*_j$ so that each $f|A_k$ represents a trivial simplicial loop in G (i.e. $f(BdA_k)$ is a single vertex and $f|A_k$ is a homotopically trivial loop) and so that when the syllables corresponding to the intersections $A_k \cap C_i$ are deleted, the words r_j becomes freely reduced. By doing a little subdivision of G if necessary, and of the triangulation of BdJ, and then cutting back slightly on the arcs A_k, we may suppose that these arcs A_k are disjoint and contain no basepoint $*_j$ and we may suppose that the 0-spheres $f(BdA_k)$ are mapped to distinct vertices of G. Note that we cannot have a point of any BdA_k on any C_i.

II. *Factoring the trivial loops through loops in trees.* For each A_k, the loop $f|A_k$ can be factored simplicially through a loop in a tree. One example of such a factoring comes from lifting $f|A_k$ to the universal covering $\tilde{G}$ of G. The image of the lifting of $f|A_k$ is a tree. By a maximality argument, we can find, among all factorings through trees, one (not necessarily unique),

$$\varphi_k: A_k \to T_k \qquad \varphi_k(BdA_k) = o_k$$
$$\psi_k: T_k \to G \qquad \psi_k \circ \varphi_k = f|A_k$$

so that φ_k takes 1-simplexes to 1-simplexes and so that for each 1-simplex σ of T_k, there are exactly two 1-simplexes of A_k that are mapped onto σ. The construction is illustrated in Fig. 1. To see that the factorizations are not unique, the reader should construct, for a suitable mapping, the factorizations corresponding to the cancellation sequence,

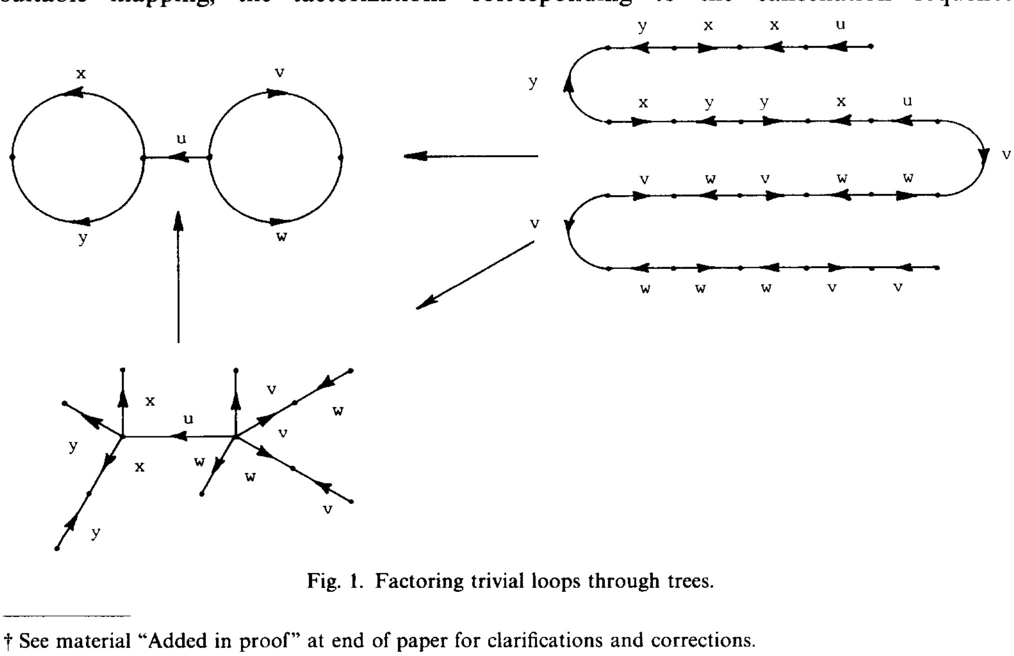

Fig. 1. Factoring trivial loops through trees.

† See material "Added in proof" at end of paper for clarifications and corrections.

$x^+x^-x^+x^-x^+x^-$. Among the possibilities for the tree are: a triod of three 1-simplexes and a simplicial arc with three 1-simplexes. For different factorizations, the endpoints of A_k go to either an endpoint or an interior vertex.

Consider the union, disjoint union, and disjoint union with identification,

$$A = \cup A_k$$
$$T = \amalg T_k$$
$$G(1) = (G \amalg T)/\sim$$
$$\sim : o_k = \psi_k(o_k) \text{ for each } k$$

Define maps $\Phi_1 : S \to G(1)$ and $\Psi_1 : G(1) \to G$ by,

$$\Phi_1 | (S \setminus A) = f | (S \setminus A)$$
$$\Phi_1 | A_k = \varphi_k \text{ for each } k$$
$$\Psi_1 | G = \text{identity}$$
$$\Psi_1 | T_k = \psi_k \text{ for each } k$$

III. *Linked cancellation segments*: Define a function β: $T \to [0, \infty)$, by means of induction, starting with $\beta(o_k) = 0$ for each basepoint o_k. If at some stage, σ is a 1-simplex of T with vertices u and v where β has been defined on u but not on v, then set $\beta(v) = \beta(u) + 1$ and extend by linearity to all of σ. For each point $t \in T$, define a subtree $T(t)$ to be the largest subtree of T containing t such that $\beta(T(t)) \subset [\beta(t), \infty)$; that is, $T(t)$ is the union of all branches of T extending from t. Let $A(t)$ denote the arc $\Phi_1^{-1}(T(t))$.

Orient the edges of T in the direction of increasing β. Consider the set Σ of pairs of distinct edges of T, (σ, σ'), such that $\beta(\sigma) = \beta(\sigma')$ and $\Psi_1(\sigma) = \Psi_1(\sigma')$. Define a subset Σ_0 of Σ to consist of those pairs (σ, σ') where σ and σ' map with opposite orientations to the same simplex of G under Ψ_1. Consider a pair of (cancellation) segments $(A(b(\sigma)), A(b(\sigma')))$ where $(\sigma, \sigma') \in \Sigma$. We say that the pair is *linked* if the two 0-spheres $Bd(A(b(\sigma)))$ and $Bd(A(b(\sigma')))$ are linked in the 1-sphere $f^{-1}(b(\tau))$ where $\tau = \Psi_1(\sigma) = \Psi_1(\sigma')$. See Fig. 10. We say that S has *linked cancellation segments* if some pair of cancellation segments is linked as defined above. The theorem below gives the condition under which our free reduction construction can be completed.

THEOREM A. *If S has no linked cancellation segments, then there is a 3-deformation $K \underset{M \times [0,1]}{\nearrow\!\searrow^3} L$ of K in $M \times [0, 1]$ to a new 2-complex L (not necessarily in $M \times 0$) so that the deformation holds fixed the 1-skeleton G of K together with $K \setminus J$ and so that each 2-cell e_j of K is converted to a 2-cell e_j' that reads the free reduction of the relator word read by e_j.*

The lemma below shows that the orientation condition is essential for the linking of cancellation segments.

LEMMA 3.1. *Let $(\sigma, \sigma') \in (\Sigma \setminus \Sigma_0)$.*
Then the corresponding pair of cancellation segments $(A(b(\sigma)), A(b(\sigma')))$ is not linked.

Proof. The proof is basically by an homology argument. The arcs $A(b(\sigma))$ and $A(b(\sigma'))$ begin and end on the simple closed curve $C = f^{-1}(b(\tau))$ where both σ and σ' map to τ under Ψ_1. These arcs can be completed to 1-cycles by adding arcs in C. Because these 1-cycles represent elements of the kernel $H_1(BdJ) \to H_1(J)$, each of the arcs $A(\sigma)$ and $A(\sigma')$ must leave and return on the same side of C. The orientation condition forces the distinguished side of C to be the same in the two cases; thus if the boundaries of the two arcs $A(b(\sigma))$ and $A(b(\sigma'))$ are linked in C, then the two 1-cycles just constructed have odd intersection number on the surface BdJ. But this contradicts the fact that the kernel of the homomorphism $H_1(BdJ) \to H_1(J)$ has a basis on which the intersection form is trivial. Figure 10 reveals how the removal of the orientation condition creates the opportunity for linking. The subset Σ_0 of Σ will require careful attention in subsequent stages of the proof of Theorem A.

Continuation of the construction. From this point on we will assume that S has no linked cancellation segments.

IV. *Another graph.* We would like to use the maps $r: J \to G$ and Ψ_1: $G(1) \to G$ and a pullback construction to create a 3-complex $J(1)$ almost like the handlebody J. Applying the pullback construction to $K \cap J$, we would be able to find a complex $K(1) \subset J(1)$ to which we could apply visual deformations. These deformations would model our anticipated deformations in $M \times [0, 1]$. What is needed, in order for us to do this, is not that $J(1)$ be a 3-manifold but rather that it admit a partitioning into 3-manifold (ball) pieces. The graph $G(1)$ is not well suited to this purpose, for when Ψ_1 is not a local homeomorphism at a vertex v, then the pullback loses all resemblance with a 3-manifold near v. We will replace $G(1)$ with a new graph $G(2)$ to get around this problem. We will define $G(2)$ by passing to the second barycentric subdivision $G''(1)$ of $G(1)$, adding some edges, and introducing an identification so that for each vertex v of T (before subdivision) the vertex star $N(v, G''(1))$ is made to look like $N(\Psi_1(v), G'')$.

Note that the subdivision $G''(1)$ of $G(1)$ causes the map Ψ_1: $G''(1) \to G''$ to be simplicial. Consider the disjoint union,

$$\text{Stars} = \amalg\, v \times (N(\Psi_1(v), G''))$$
$$(v \text{ a vertex of } T)$$

Define a new graph $G(2)$ by,

$$G(2) = (G''(1) \amalg \text{Stars})/\sim$$

where $\sim$ is defined by,

$$(v, y) \in v \times N(\Psi_1(v), G'') \text{ and } x \in G''(1)$$
$$x \sim (v, y) \text{ if}$$
$$x \in N(v, G''(1)) \text{ and } \Psi_1(x) = y.$$

By transgression (see Dugundji [15, VI, Th. 3.2]), the quotient map induced by $\sim$ in turn induces new maps,

$$\Phi_2: S \to G(2)$$
$$\Psi_2: G(2) \to G''$$

so that $\Psi_2 \circ \Phi_2 = \Psi_1 \circ \Phi_1$. The map Φ_2 is precisely Φ_1 followed by the quotient map; the map Ψ_2 is obtained by taking any preimage in $G(1)$ of a point under the the quotient map and then applying Ψ_1 to that preimage. For a 1-simplex σ of $G(1)$, we will, by applying the quotient map induced by $\sim$, also regard σ as a subset of $G(2)$; although σ becomes here the union of four 1-simplexes.

V. *A pullback construction.* Recall the retraction r associated with the mapping cylinder structure C_f on J. Define a pullback diagram,

$$\begin{tikzcd} J(2) \arrow[r, "r'"] \arrow[d, "\Psi'_2"] & G(2) \arrow[d, "\Psi_2"] \\ J \arrow[r, "r"] & G''. \end{tikzcd}$$

Let $BdJ(2)$ denote the subset $(\Psi'_2)^{-1}(BdJ)$. Now $J(2)$ is given a mapping cylinder structure by a map f' that is the restriction of r' to $BdJ(2)$. The mapping cylinder coordinates for $C_{f'}$ are given by the retraction r' and a map λ': $J(2)\to[0, 1]$ associated with λ.

We define a homeomorphic copy $S(2)$ of S in $BdJ(2)$:

$$S(2)=\{x\in Bd(J(2))|\Psi'_2(x)\in S \ \& \ \Phi_2\circ\Psi'_2(x)=r'(x)\}.$$

Let $E(2)$ denote the restricted mapping cylinder $C_{f'}|S(2)$ and regard $E(2)$ as a subpolyhedron of $J(2)$. Define a 2-complex $K(2)$ to be the adjunction space formed from $E(2)$ and $Cl(K\setminus J)$ by the identification of S and $S(2)$ via the homeomorphism $\Psi'_2|S(2)$. The 1-skeleton of $K(2)$ is $G(2)$ and the 2-cells are the components of $K(2)\setminus G(2)$. Notice that when the maximal tree is chosen properly (add T to the maximal tree associated with $\mathscr{P}_K$), the presentation $\mathscr{P}_{K(2)}$ associated with $K(2)$ is obtained from $\mathscr{P}_K$ by freely reducing the relators.

For each vertex w of G, let $B(w)$ denote the 0-handle (*ball*) $r^{-1}(N(w, G''))$, and for each 1-simplex σ of G, let $B(\sigma)$ denote the 1-handle (*beam*) $r^{-1}(N(b(\sigma), G''))$. These handles are 3-cells and describe a handle decomposition of J. The pullback $J(2)$ carries a rather similar structure. For each vertex v of $G(1)$, there is a 3-cell $B(v)$ (henceforth called a *ball*) that contains v and maps homeomorphically to $B(\Psi_2(v))$ under Ψ'_2. For each 1-simplex σ of $G(1)$, there is a 3-cell $B(\sigma)$ (henceforth called a *beam*) containing $b(\sigma)$ that maps homeomorphically to $B(\Psi_1(\sigma))$ under Ψ'_2. These balls and beams account for all of $J(2)$, and the only places where $J(2)$ fails to be a 3-manifold handlebody is at the junction of balls and beams where it is possible for several different beams to join to a ball along a common disk. See Fig. 7.

We will refer to the intersections of balls and beams in $J(2)$ as the *faces* of the balls. If v is a vertex of a 1-simplex σ of $G(1)\setminus G$, then the disk $B(v)\cap B(\sigma)$ will be called an *upper face* of $B(v)$ if $v\notin T(b(\sigma))$ where $T(b(\sigma))$ is the tree defined in Step III. Similarly the disk $B(v)\cap B(\sigma)$ will be called the *lower face* provided $v\in T(b(\sigma))$. Notice that a ball can have many upper faces but only one lower face.

VI. *Mapping* $G(2)$ *to* $G\times[0, 1]$. The object of the free reduction construction is to try to use the extra dimension in $M\times[0, 1]$ to deform K near G into a copy of the restricted mapping cylinder $E(2)$ so that the map $\Psi'_2|E(2)$ is represented by the projection map proj_1 from $M\times[0,1]$ to M. If we could achieve this, then our new 2-complex would be, up to isomorphism, $K(2)$ and so would have the desired freely reduced readings in $\mathscr{P}_{K(2)}$. To feel really secure we could even try to detach, through a further deformation, the copy of $K(2)$ from the 1-skeleton along the copy of T in $G(2)$ leaving G as the new 1-skeleton; although this would be unnecessary.

Now we cannot achieve the deformation indicated in the preceding paragraph because we cannot lift Ψ_2 to an embedding of $G(2)$ into $G\times[0, 1]$ that is converted to Ψ_2 under the projection map of $G\times[0, 1]$ to G. But we can deform K as above into the union of $G\times[0, 1]$ and an immersion of $E(2)$ so that $\Psi'_2|E(2)$ is represented by projection, and we can carry out the detachment from the immersed copy of T in conjunction with a collapse of $G \times [0, 1]$. This detachment now becomes a necessary operation because of the singularities of the immersion. This approach to deformation will turn out to be sufficient for our needs. We start by describing an immersion θ_2 of $G(2)$ into $G \times [0, 1]$.

A first approximation to θ_2. Define θ_2 on G by $x \to (x, 0)$. Define an approximation to θ_2 on the vertices u of T by $(\Psi_1(u), \beta(u)/2B)$ where B is the maximum value of β on T. Extend this approximation to the neighborhoods $N(u, G(2))$ by $x \to (\Psi_2(x), \beta(u)/2B)$. For each 1-simplex σ of T, the approximation has now been defined on $N(\dot{\sigma}, G(2)) \cap \sigma$. Extend to the rest of σ by linearity. This completes the first approximation to θ_2.

By an induction argument involving some small adjustments to the first approximation, we will be able to eliminate all singularities of θ_2 except those that arise because θ_2 causes pairs of 1-simplexes to cross in $G \times [0, 1]$. But these singularities occur for precisely the pairs (σ, σ') in Σ_0. Define as follows a lexicographic type ordering on the vertices of T, first on $\beta^{-1}(0)$, then extending to $\beta^{-1}(\{0, 1\})$ and so on: Order in some arbitrary fashion the vertices of G. This induces an ordering on $\beta^{-1}(0)$. Suppose that for some integer k, the ordering has been extended up to $\beta^{-1}(\{0, \ldots, k-1\})$. Use the following rule to extend the ordering to $\beta^{-1}(\{0, \ldots, k\})$:

$$u < v \text{ if}$$

(a) $\beta(u) < \beta(v)$ and $\beta(v) = k$, or $\beta(u) = \beta(v) = k$ and
(b) u and v are vertices of 1-simplexes σ_u and σ_v of T whose other vertices are $u' \in \sigma_u$ and $v' \in \sigma_v$ where $u' < v'$, or $\beta(u) = \beta(v) = k$, u and v are vertices of different 1-simplexes σ_u and σ_v that share a common vertex at the other end, and
(c) $\Psi_1(u) < \Psi_1(v)$.

This rule still fails to distinguish sets of vertices $\{u_i\}$ with $\beta(u_i) = k$ when $\Psi_1(u_i)$ is the same for all i and in addition the u_i's are vertices of different 1-simplexes that all share a common vertex, say v at the other end ($\beta(v) = k-1$). For such sets of vertices order the vertices in some arbitrary fashion.

Define a second approximation to θ_2 by adjusting the first approximation slightly on the vertices of $T \setminus G$ so that $\mathrm{proj}_2 \circ \theta_2(u) < \mathrm{proj}_2 \circ \theta_2(v)$ if $u < v$. Do this so that the vertices of T map to distinct irrational levels in the second coordinate, and so that the singularities of θ_2, where pairs of simplexes are made to cross, are mapped to distinct irrational levels that are also distinct from the levels of the vertices. Finally adjust the second coordinate function proj_2, if necessary, on the neighborhoods $N(u, G(2))$ where u is a vertex of T so that proj_2 is constant on these neighborhoods. We assume that these adjustments are sufficiently fine so that the conditions below are satisfied:

(1) $\Psi_2 = \mathrm{proj}_1 \circ \theta_2$ and $\theta_2 | G$ is given by $x \to (x, 0)$.
(2) For each 1-simplex σ of $G(1)$, the map θ_2 is linear on $\sigma \setminus N(\dot{\sigma}, G(2))$.
(3) For each vertex u of $G(1)$, the map $\mathrm{proj}_2 \circ \theta_2$ is constant on $N(u, G(2))$.
(4) If u and v are different vertices of the same 1-simplex σ of $G(1) \setminus G$ with $v \in T(u)$, then $\mathrm{proj}_2 \circ \theta_2(v) > \mathrm{proj}_2 \circ \theta_2(u)$.
(5) For each pair $(\sigma, \sigma') \in \Sigma_0$, the map θ_2 causes σ and σ' to cross at some point $\theta_2(a(\sigma, \sigma')) = \theta_2(a(\sigma', \sigma))$ in $\Psi_1(\sigma) \times [0, 1] = \Psi_1(\sigma') \times [0, 1]$ where $a(\sigma, \sigma') \in \sigma \setminus N((\dot{\sigma}, G(2))$ and $a(\sigma', \sigma) \in \sigma' \setminus N((\dot{\sigma}', G(2))$.
(6) The singularities in (5) are the only singularities of θ_2.
(7) Under $\mathrm{proj}_2 \circ \theta_2$, the vertices of $G(1) \setminus G$ map to distinct, irrational levels, and the singular pairs $(a(\sigma, \sigma'), a(\sigma', \sigma))$ of (5) map to distinct, irrational levels that are also distinct from the levels of the vertices.

There is an immediate extension of θ_2 to a map Θ_2: $J(2) \to J \times [0, 1]$ given in mapping cylinder coordinates by $(x, t) \to (\Psi'_2(x, t), \mathrm{proj}_2 \circ \theta_2 \circ f'(x))$. The singularities of Θ_2 come from those of θ_2 by crossing with a disk: For each pair (σ, σ') in Σ_0, the corresponding pair of beams $(B(\sigma), B(\sigma'))$ is made to intersect in a disk in $J \times [0, 1]$. See Fig. 8. However the singularities of $\Theta_2|E(2)$ are contained in $G(2)$ and are simply those of θ_2.

VII. *Deforming K to a new 2-complex.* Define a map Ξ from $K(2)$ to $M \times [0, 1]$ as follows: On $K(2) \cap J^{1/2}(2)$ take Ξ to be Θ_2. On $Cl(K \setminus J)$ take Ξ to be the identity map. Now splice the two partial maps together with a homeomorphism by using vertical deformations so that on $K(2) \cap J(2)$, we have $\Psi'_2 = \mathrm{proj}_1 \circ \Xi$.

Let $L(2)$ denote the image $\Xi(K(2))$, and let F denote the fence $G \times [0, 1]$. By using a sequence of vertical cylindrical expansions and collapses, guided by the map Ξ, one verifies the following lemma:

LEMMA 7.1. *There is a 3-deformation*

$$K \nearrow K \cup F \quad \searrow L(2) \cup F$$

in $M \times [0, 1]$ that holds G fixed.

VIII. *Outline for remainder of construction*: We will define four sequences of modifications in $M \times [0, 1]$ fixing G:

$$\begin{aligned}
G(2) &= G(2,0) \searrow \dots \searrow G(2,k) \searrow \dots \searrow G(2,n) = G \\
K(2) &= K(2,0) \nearrow\!\!\searrow^3 \dots \nearrow\!\!\searrow^3 K(2,k) \nearrow\!\!\searrow^3 \dots \nearrow\!\!\searrow^3 K(2,n) \\
L(2) &= L(2,0) \to \dots \to L(2,k) \to \dots \to L(2,n) \\
F &= F(0) \searrow \dots \searrow F(k) \searrow \dots \searrow F(n) = G \times 0
\end{aligned}$$

so that at every stage k, the following conditions are met:

(1) The 1-skeleton of $K(2, k)$ is contained in $G(2)$ and $\mathscr{P}_{K(2,k)} \equiv \mathscr{P}_{K(2)}$ (same generators, relators identical as words).
(2) The 1-skeleton of $L(2, k)$ is contained in $F(k)$.
(3) $L(2, k) = (L(2) \setminus J^{1/2}(2)) \cup \Theta_2(K(2, k) \cap J^{1/2}(2))$.
(4) $L(2, k) \cup F(k) \nearrow\!\!\searrow^3 L(2, k+1) \cup F(k+1)$ in $M \times [0, 1]$.
(5) The singularities of $\Theta_2|K(2, k) \cap^{1/2} J(2)$ are contained in the preimage of $F(2, k) \setminus G$.

The map Θ_2 will be non-singular on $K(2, n) \cap J^{1/2}(2)$; hence, by Lemma 7.1, we will have succeeded in deforming K to $L = L(2, n)$ so that the presentation $\mathscr{P}_L$ is $\mathscr{P}_{K(2)}$ and so will have been obtained by freely reducing the relators of $\mathscr{P}_K$. The role of the collapse $G(2) \searrow G$ in all this will be to keep the other deformations in harness. The unlinked singularities condition will be necessary so that we can turn deformations of $K(2)$, defined in the abstract pullback $J(2)$, into deformations of $L(2) \cup F$.

A note on indexing. The stages of the deformations will be indexed by lexicographically ordered tuples of natural numbers. If we need further refinement of some steps, we will simply increase the size of the tuple. Because of the indexing needs, there will be many inactive stages for the individual deformations. Finally, we will be describing modifications of objects that intersect: $L(2)$ and F. It will be important that at appropriate stages the modifications be compatible. At one of these critical stages a modification of one of $L(2)$ or F will be labeled # if it will have to be absorbed by a modification of the other.

IX. *Further notation*

Side collapses. One technique we will use repeatedly in collapsing the fence F is a collapsing from the side to go underneath pieces of the graph $\theta_2(G(2))$ that remain at a certain stage. Let $0 \leq a < b \leq 1$ be real numbers and let X be a graph in $G \times [a, b] \subset G \times [0, 1]$ that maps 1–1 *into* G under proj_1. Let X collapse to a subgraph X_0. By the $[a, b]$-*side collapse under* X (*corresponding to the collapse* $X \searrow X_0$) we mean the collapse (see Fig. 2) $Y \searrow Y_1$ where,

$$Y = \{(y, t) \in G \times [0, 1]\text{: for some } (x, s) \in X,\ y = x \text{ and } t \leq s\}$$

$$Y_0 = \text{same as } Y \text{ except require that } (x, s) \in X_0$$

$$Y_1 = Y_0 \cup ((\text{proj}_1(X)) \times [0, a]) \cup X$$

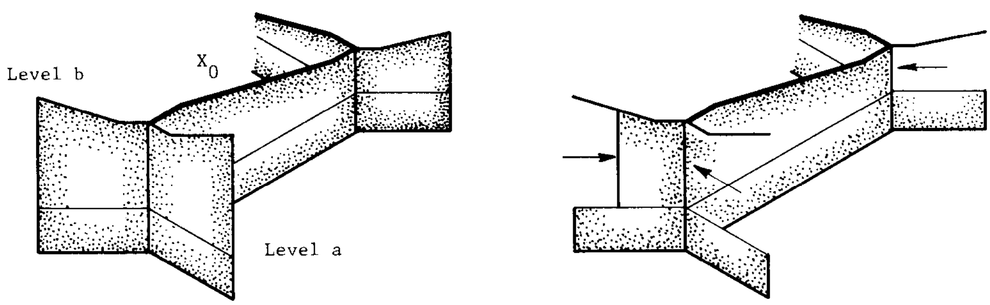

Fig. 2. A side collapse.

There are some variations on this definition that we need to discuss: For the first variation let X be a graph in $G \times [0, 1]$ with no vertical edges but that does not necessarily map 1–1 to $G \times [0, 1]$ under proj_1. We say that a collapse $X \searrow X_0$ is *monotonically decreasing* if each edge of $X \backslash X_0$ maps 1–1 into $[0, 1]$ under proj_2, each edge to be collapsed collapses to its bottom vertex, and finally no point of $X \backslash X_0$ lies below a point of X_0. Under these conditions, by the *side collapse under* X (*corresponding to the collapse* $X \searrow X_0$) we mean the collapse $Y \searrow Y_0$ where Y and Y_0 are defined exactly as before. For the second variation let X and X' be straight line intervals in $G \times [a, b]$ that map 1–1 into G under proj_1, and suppose that $X \cap X'$ is a single point interior to both X and X'. Let X_0 and X'_0 be the lower endpoints of X and X' respectively, and suppose that they are both at the level a of $G \times [0, 1]$. Define shadow polyhedra Y and Y' as before. By the *modified side collapse under* X and X' (*corresponding to the collapses* $X \searrow X_0$ and $X' \searrow X'_0$) we mean the collapse $(Y \cup Y') \searrow (Y \cap Y') \cup (X \cup X')$. We will call the polyhedron $(Y \cap Y') \cap (G \times [a, b])$ the *triangle* (*under the intersection*).

Let n be a natural number sufficiently large so that for any natural number k, the interval $[(k-2)/n, k/n]$ contains the image under $\text{proj}_2 \circ \theta_2$ of at most one of the following items:

(1) a singular pair of points $(a(\sigma, \sigma'), a(\sigma', \sigma))$ where $(\sigma, \sigma') \in \Sigma_0$,
(2) a vertex of $G(1) \backslash G$.

Also it is assumed that n is so large that when the images under θ_2 of two edges σ and σ' ($(\sigma, \sigma') \in \Sigma_0$) are made to cross say in some $\tau \times [(k-1)/n, k/n]$ (τ a 1-simplex of G), then there is some subinterval Z of τ such that the box $Z \times [(k-1)/n, k/n]$ contains the intersections $\theta_2(\sigma \cup \sigma') \cap \tau \times [(k-1)/n, k/n]$ but does not contain the image under θ_2 of any other edge of G_1.

Some pieces of $G(2)$. Consider, for some natural number k, a component X of $\theta_2^{-1}\circ \text{proj}_2^{-1}([1-(k+1)/n, 1-k/n])$. If X contains no vertex of $G(1)\backslash G$, then we call X a *rod*. If X contains a vertex, say u, then we give names to certain pieces of X as indicated below:

Let $z = \text{proj}_2\circ\theta_2(u)$. The closures of the components of $X\cap\theta_2^{-1}\circ\text{proj}_2^{-1}((z, 1-k/n])$ we call *upper rods*, and the closure of the unique component $X\cap\theta_2^{-1}\circ\text{proj}_2^{-1}([1-(k+1)/n, z))$ we call a *lower rod*. We call the component of $X\cap\theta_2^{-1}\circ\text{proj}_2^{-1}(z)$ containing u the *wheel about u*. Notice that a rod intersects a wheel in at most one point. There are, in the wheel, certain maximal arcs with endpoints u. These are called *spokes*. The ones that connect to upper rods are called *upper spokes*, the one connecting to a lower rod is called the *lower spoke*, and any remaining spokes are called *dummy spokes* (these do not come from $G(1)$ but rather from the passage to $G(2)$). The terms *upper* and *lower* refer to the values of proj_2. We will speak of upper and lower endpoints of rods in the same way.

Some pieces of $J^\varepsilon(2)$. Let $B(\sigma)$ be a beam in $J(2)$. A 3-dimensional component of $B^\varepsilon(\sigma)\cap\Theta_2^{-1}\circ\text{proj}^{-1}([1-(k+1)/n, 1-k/n]$ will be called a *beam segment*. The boundary of the beam segment is made up of a *lateral surface* (an annulus) and *upper* and *lower faces* corresponding to $\text{proj}_2^{-1}(1-k/n)$ and $\text{proj}_2^{-1}(1-(k+1)/n)$. A corresponding intersection $B^\varepsilon(v)\cap\Theta_2^{-1}\circ\text{proj}^{-1}([1-k/n])$ involving a ball $B^\varepsilon(v)$ in $J(2)$ will be called a *ball section*.

The Deformation Table contains a summary of all deformations with the proper indexing. These deformations are described individually in the next four sections. Also, in the Appendix, a family of mapping cylinder deformations is described. These deformations are used in the third and fourth sequences in the modifications.

X. *The modifications, Part* (1): $G(2)\searrow G$.

Set,

$$\begin{aligned} G(2,k) = G(2,k,0) &= G(2)\cap\theta_2^{-1}\circ\text{proj}_2^{-1}([0, 1-k/n]) \\ X(k) &= \theta_2^{-1}\circ\text{proj}_2^{-1}([1-(k+1)/n, 1-k/n]), \text{ and} \\ X_0(k) &= \theta_2^{-1}\circ\text{proj}_2^{-1}(1-(k+1)/n). \end{aligned}$$

The collapses $G(2,k,0)\searrow\ldots\searrow G(2,k,l)\searrow G(2,k,l+1)\searrow\ldots\searrow G(2k,40) = G(2,k+1,0)$ are as follows. The collapses are inactive except in some of the stages $G(2,k,2j+1)\searrow G(2,k,2j\to 2)$, and for these stages the activity if any, is described below.

$j = 0$–1, 3–5, 7, 9–11, 13–15, 18: Inactive.

$j = 2$. For any pair of rod components of $X(k)$ that contains a singular pair $(a(\sigma,\sigma'), a(\sigma',\sigma))$, collapse the pair of rods to their lower endpoints.

$j = 6$. Except for upper and lower rods, collapse any remaining rod components of $X(k)$ to their lower endpoints.

$j = 8$. Collapse any upper rods of $X(k)$ to their lower endpoints.

$j = 12$. Collapse any dummy spokes of $X(k)$ to their hub vertex.

$j = 16$. Collapse any upper spokes of $X(k)$ to their hub vertex.

$j = 17$. Collapse any lower spoke of $X(k)$ to its intersection with the corresponding lower rod.

$j = 19$. Collapse any lower rod to its lower endpoint.

XI. *The modifications, part* (2): $F\searrow G\times 0$.

Set $F(k) = G\times[0, 1-k/n]$. The collapses $F(k,0)\searrow\ldots\searrow F(k,l)\searrow F(k,l+1)\searrow F(k,40) = F(k+1,0)$ are as follows. The only active stages of the collapse are some of the stages $F(k, 2j) \searrow F(k, 2j+1)$, and these are described below:

$j = 0$: Collapse down to the union of $F(k+1)$ and the shadow under $\theta_2(G(2, k))$.

$j = 1$: Do a modified $[1-(k+1)/n, 1-k/n]$-side collapse under the images of any pair of rods that intersect.

$j = 2, 13, 14, 15$: Inactive.

$j = 3$: Collapse the images of any singular pair of rods down to their point of intersection.

$j = 4$: Collapse the triangle under the intersecting rod images to the base of the triangle located at the level $1-(k+1)/n$.

$j = 5$: Except for images of upper and lower rods, do a side collapse under the remaining (non-singular) rods. Depending on the picture this may need to be of the first variation type defined in Step IX.

$j = 6$: Collapse the images of any rods involved in stage 5 to their lower endpoints.

$j = 7$: Do a side collapse under the images of any upper rods. If there are several upper rods joined together then this side collapse must be of the first variation type defined in Step IX.

$j = 8$: Collapse the images of any upper rods to their lower endpoints.

$j = 9$: Do a side collapse under the images of any upper spokes.

$j = 10$: Do a side collapse under the images of any dummy spokes.

$j = 11$: Do a side collapse under the image of any lower spoke.

$j = 12$: Collapse the images of any dummy spokes to the hub vertex image.

$j = 16$: # Collapse the images of any upper spokes to the hub vertex image.

$j = 17$: # Collapse the image of any lower spoke to the image of its intersection with the lower rod.

$j = 18$: Do a side collapse under the image of any lower rod.

$j = 19$: Collapse the image of any lower rod to its lower endpoint.

XII. *The modifications, part* (3): $K(2) \nearrow\!\searrow^3 K(2, n)$.

In the deformation, we start with $\varepsilon = 1/4$ and deform the appropriate part of $K(2, \ldots)$ in $J^\varepsilon(2)$ using, if necessary, the space $J^{2\varepsilon}(2)\backslash J^\varepsilon(2)$ to splice a deformation with the identity. Then each time we advance a stage in the lexicographic ordering, we replace ε by $\varepsilon/2$ and proceed further. The deformations $K(2, k) = K(2, k, 0) \nearrow\!\searrow^3 \ldots \nearrow\!\searrow^3 K(2, k, l) \nearrow\!\searrow^3 K(2, k, l+1) \ldots \nearrow\!\searrow^3 K(2, k, 40) = K(2, k+1, 0)$ are as follows. The only active stages are certain of the stages $K(2, k, 2j+1) \nearrow\!\searrow K(2, k, 2j+2)$ and these are described below. The marks # refer to modifications of $L(2)$ since no deformations of the abstract $K(2)$ will be absorbed.

There are certain mapping cylinder moves used for this section. They are described in the Appendix with figures to illustrate each move. The purpose of these moves is to peel back part of $K(2, k)$ from $G(2)$ while maintaining the property that the 1-skeleton of $K(2, k)$ is contained in $G(2)$. The part to be peeled back is the part of $K(2, k)$ in $J^{1/2}(2) \cap \Theta_2^{-1} \circ \mathrm{proj}_2^{-1}((1-(k+1)/n, 1-k/n])$; although isolated points of the deformed $K(2, k)$ may be allowed to reappear in the intersection just mentioned. The activity of a move will be isolated to the part of $J^\varepsilon(2)$ contained in $\Theta_2^{-1} \circ \mathrm{proj}_2^{-1}([1-(k+1)/n, 1-k/n])$. If a move is mentioned at a certain stage, it is understood that the move is applied only if there is a subcomplex of the appropriate type in the inverse image just mentioned; otherwise that stage is considered to be inactive.

Here is our situation. The preimage $\Theta_2^{-1} \circ \mathrm{proj}_2^{-1}([1-(k+1)/n, 1-k/n])$ consists of a collection of components: beam sections, at most one ball (corresponding to a vertex) with upper and lower beam sections attached to it (possibly junctions of beams attached along upper faces of the ball), and at most one pair of beam sections that are made to intersect by Θ_2. The 1-skeleton of $K(2, k, 2j+1)$ is always $G(2, k, 2j+1)$. There may be, at various stages, other points of $G(2)$ in $K(2, k, 2j+1)$, but these will be isolated points and disjoint from $\theta_2^{-1}(F(k, 2j+1))$.

$j = 0$: *inverting a trough end.* Stages $j = 1, 5, 7$ may require some preliminary trough end inversions before the moves can be effected. Use this stage to make all the necessary trough end inversions.

$j = 1$: *moving down a beam.* This step is reserved for the moves down any pair of beam segments that are made to intersect by Θ_2. When we pass to the modifications of $L(2, k)$ these moves will become the moving down intersecting beams move, and any choices that we have to make on the ordering of the moves, beam diameters, and preliminary trough end inversions will be assumed to be retroactive to stages $j = 0, 1$ here.

$j = 2, 6, 8$: *# graph collapses.* These are graph collapses designed to pull residual parts of K away from $G(2)$. The collapses are identical with the corresponding graph collapses defined in Step X.

$j = 3, 4, 9$–$11, 16, 17$. Inactive.

$j = 5$: *moving down a beam.* Use this step to move down all beam segments that are not covered by the stage $j = 1$ and do not intersect a ball (these cases are covered by stages $j = 7$, 18).

$j = 7$: *moving down a junction of beams*: Use this step to move down beams that attach to upper faces of a ball. If there is just one beam attaching to an upper face, treat as a junction of one beam. As indicated in the Appendix there is an ordering condition that must be satisfied in order to move down a junction of beams. Trough end inversions are used to effect this ordering condition. The proof that the ordering condition can be achieved involves induction and an argument very similar to the proof of Lemma 3.1.

The moves for $j = 5$ and $j = 7$ pull the complex $K(2, k, 2j+1)$ away from the corresponding rod sections leaving the rods to be disposed of by later moves ($j = 6, 8$).

$j = 12$: *# Collapsing dummy spokes.* Collapse any dummy spokes to their hub vertex. The dummy spokes are always a collapsible part of the complex $K(2)$.

$j = 13$: *preliminary cone moves.* Whenever we have moved down junctions of beams, even in the case of junctions of only one beam, the complex $K(2, k, 2j+1)$ intersects the upper faces of the corresponding ball in cones. For each upper face the cone is over a disjoint union of arcs in the boundary of the upper face. The cone point is of course the intersection of one of the upper rods with the upper face. Now the ball and the remainder of the intersection of the complex $K(2, k, 2j+1)$ with the ball carry a cone structure from the vertex of the ball. Use this cone structure on the ball and the vertex to first expand the intersection to include the cones from the vertex over the upper face intersections. Then collapse from the upper face intersections towards the vertex to collapse these 3-dimensional cones back to the cones over arcs in the upper face boundaries.

At this point the intersection of the complex with the ball has the form of a cone from the vertex over a singular simple closed curve on the boundary of the ball. The singularities of the curve come from pushing sections together into tangent intersections. See the curve in Fig. 6a. The complex intersects the various joining beams in the surfaces we called trough covers. Adjust these trough covers as they move up from the upper faces so that near but not on the upper faces they are pushed slightly into the exteriors of the beams and at the same time are pushed so that the projection map is 1–1 near the upper faces. By using the product structures on the beams we can now bring the part of the trough covers near the upper faces into the 3-dimensional level cross sections at the level of the ball. Things now appear as in Fig. 6a; although there may be more surfaces pinched together along upper faces.

$j = 14$: *desingularizing a cone.* This move is used whenever we have just used a moving down a junction of two or more beams move in the stage $j = 7$ and have used the previous stage to get a complex meeting the corresponding ball in a cone over a singular simple closed curve.

$j = 15$: *pushing out a cone.* This move is used to free from $G(2)$ the part of $K(2, k)$ inside a ball.

$j = 18$: *moving down a beam.* If there is any beam segment left that attaches to a lower face then move down this beam segment now. Such a beam segment would correspond to a lower rod.

$j = 19$: *collapsing lower rods.* Collapse the image of any lower rod to its lower endpoint.

XIII. *The modifications, part* (4):

$$F(k, \ldots) \cup L(2, k, \ldots) \nearrow\!\searrow^3 F(k+1, \ldots) \cup L(2, k+1, \ldots).$$

First we describe the modifications of $L(2)$. Define $L(2, k, l)$ to be the union $(E(2)\setminus J^{1/2}(2)) \cup \Theta_2(K(2, k, l) \cap J^{1/2}(2))$. The active stages in the modifications of $L(2)$ are precisely the same as the active stages in the modifications of $K(2)$. The only stages needing any explanation are the stages $L(2, k, 2j+1) \to L(2, k, 2j+2)$ with $j = 1, 2$. All other stages are deformations.

$j = 1$: *moving down intersecting beams.* It follows from the non-linking condition on the cancellation segments that for intersecting beams in $L(2, k, 2j+1)$, if we slide the upper trough ends down to the disk of intersection, we get one of the two allowable cases (a) or (b) of the moving down intersecting beams move. See Figs 8 and 9. With the care in the ordering of the deformations and with the conditions on the beam thickness described in the Appendix, the deformation will induce a corresponding deformation on $L(2, k, 2j+1)$ such that the singularities condition (5) in Step VIII is met. Note that the central rods remain behind in this deformation. If they were removed then we would not have a deformation of $L(2, k, 2j+1)$.

$j = 2$: *# graph collapses.* Discard the image of the singular pair of rods. This move will be taken over by the triangle collapse part of the fence collapse.

Finally, in order to mesh together the modifications of $L(2)$ and the collapse of the fence F into a common deformation we must drop certain deformations that are accounted for twice. These are summarized in the Absorption Table and they are discussed. below.

Deformations of F that are absorbed. Except for the collapse of dummy spokes, the wheel collapses $j = 16$, 17 are dropped as they are made unnecessary by the pushing out a cone moves ($j = 15$) on $L(2, k)$. This was the reason that in stages $j = 9$, 10, 11 of the collapse of $F(k)$, we did side collapses underneath the wheel; otherwise we could not have had the cone pushing move consistent with the deformation of $F(k)$.

Deformations of $L(2, k)$ *that are absorbed.* The rod collapses $j = 2$, 6, 8, 19, and the collapse of dummy spokes $j = 12$ are dropped. These are taken care of by the corresponding collapses of $F(k)$.

With these changes, the modifications of $L(2, k)$ and $F(k)$ combine to a deformation as desired.

This completes the free reduction construction and hence the proof of Theorem A.

§4. SPLIT 2-COMPLEXES; EXTENDING THE FREE REDUCTION TECHNIQUE

The free reduction theorem in the first part of this paper was recovered from an earlier, incorrect, version [10]. In that version we asserted that if a 3-manifold with boundary, M, had a 2-spine K, and if K abstractly 3-deformed to another 2-complex L with one 0-cell, n 1-cells, and p 2-cells, then K 3-deformed in $M \times [0, 1]$ to a 2-complex with the same numbers of cells as L. An immediate consequence of this assertion was the implication that the 3-dimensional Poincaré conjecture reduced to the Andrews–Curtis conjecture for the special case of geometric presentations for the fundamental groups of homotopy 3-spheres. This implication must now be regarded as doubtful.

For any 2-complex K, let us define the *extended Nielsen genus*, $EN(\mathscr{P}_K)$, of the group presentation $\mathscr{P}_K$ to be the minimum number of generators present in any presentation that is equivalent to $\mathscr{P}_K$ under extended Nielsen transformations. It is an easy consequence of the Reidemeister–Singer theorem [22], [24], [7] and regular neighborhood theory that we can define the *extended Nielsen genus*, $EN(M)$, of a 3-manifold M to be the extended Nielsen genus of the group presentation associated with any 2-spine of M. (If M is closed remove a 3-ball first to make M a 3-manifold with boundary.) Montesinos [19] has defined the *big genus* of a 3-manifold, M, to be the minimum number of 1-cells in any 2-spine of $M \times [0, 1]$. If the result of our earlier paper could be recovered for 3-manifold spines, it would say that Montesinos big genus of a 3-manifold is no greater than the extended Nielsen genus of the 3-manifold. But the latter is bounded by the Heegaard genus of the 3-manifold for closed 3-manifolds, and is in some cases strictly less than the Heegaard genus. Montesinos [19] has exhibited strict inequality for the Boileau–Zieschang example [4] of a manifold with Heegaard genus 3 and rank 2 (rank of the fundamental group equals 2) by showing that not only is the rank of the fundamental group equal to 2, but so is the extended Nielsen genus.

Conjecture B. If M is a 3-manifold, then the Montesinos big genus of M is less than or equal to the extended Nielsen genus of M.

We outline in what follows an approach, so far unsuccessful, to proving this conjecture. First, we need to be able to apply the free reduction construction to a wider class of 2-complexes. We describe this class in the next paragraph.

Let K be a 2-complex in the interior of an orientable 4-manifold M, and let K have 1-skeleton G. Let N be a regular neighborhood of G in $\dot{M}$ that is small relative to K. By an *admissible product structure* on N we mean a pwl homeomorphism $\mu: J \times [-1, 1] \to N$ where (i) J is a second derived neighborhood of a finite graph G_0 in an orientable 3-manifold, (ii) $G = \mu(G_0)$, and (iii) $\mu^{-1}(K \cap N)$ is a collection of simplices whose vertices are all contained in the union of G_0 and $BdJ \times \{\pm 1/2\}$. Notice that $J \times [-1, 1]$ contains two distinguished copies of J that reside in the product in a diagonal like fashion: J_+ whose simplexes are joins of simplexes in $BdJ \times 1/2$ with simplexes in G_0 and J_- whose simplexes are joins of simplexes in $BdJ \times -1/2$ with simplexes in G_0.

Let K be a 2-complex in the interior of an orientable 4-manifold M. We say that K is a *split 2-complex* if the 1-skeleton of K has a small regular neighborhood with an admissible product structure. If K is a split 2-complex then each 2-cell e_i of K can intersect only one of the two pieces $\mu^{-1}(J_+)\backslash G$ of $\mu^{-1}(J_-)\backslash G$. Thus K splits as a union of two subcomplexes $K = K_+ \cup K_-$ with $K_+ \cap K_- \subset G$.

Note that, provided the orientability conditions are met, any 3-manifold 2-complex is trivially a split 2-complex in any 4-manifold whose interior contains the 3-manifold. The following theorem, taken from the corrections to the old version to this paper (see [7] and

[11]) indicates the close relationship between split 2-complexes and extended Nielsen genus:

THEOREM C. *Let K be a split 2-complex in a 4-manifold N. Then K 3-deforms in N to a new split 2-complex L with an associated group presentation of the form,*

$$\mathscr{P}_L = \{x_1, \ldots, x_m, \ldots, x_{m+n} | r_1, \ldots, r_p, \ldots, r_{p+n}\}$$

where $m = EN(\mathscr{P}_K)$, the extended Nielsen genus of the presentation $\mathscr{P}_K$, each $r_i (i \leq p)$ freely reduces to a word on the generators $x_1, \ldots, x_m$ and their inverses, and each r_{p+i} freely reduces to x_{m+i}.

If the free reduction of the presentation $\mathscr{P}_L$ could be effected by 3-deformations in N, then Conjecture B could be verified. The theorem below shows that the obstructions to free reduction for split 2-complexes can be measured by a double set of linked cancellation segment obstructions.

THEOREM D. *Let K be a split 2-complex in the 4-manifold N with $K = K_+ \cup K_-$ where $K_+ \cap K_- = G$. Then there are associated with K two sets of linked cancellation pair obstructions, one set with K_+ and one with K_-. It these both vanish, then K can be 3-deformed in N to a new 2-complex L so as to freely reduce the relators in the presentation $\mathscr{P}_K$. In particular, if in the freely reduced version of $\mathscr{P}_K$, there are k distinct free generators that are read as relators, and if these k generators do not appear in the other reduced relator words, then k faces of L can be collapsed away after the deformation above.*

Proof. Let $N \approx J \times [-1, 1]$ be a suitable small regular neighborhood of the 1-skeleton G of K. Let J_+ and J_- be the two diagonal copies of J referred to in the definition of an admissible product structure. By local collaring arguments we can find one sided collars $J_+ \times [0, 1] \to N$ and $J_- \times [0, 1] \to N$ on J_+ and J_- that intersect only in G. Now apply the free reduction construction to the part of K_+ in the image of $J_+ \times [0, 1]$ and similarly to the part of K_- in the image of $J_- \times [0, 1]$ to get the two sets of linking obstructions. If, after free reduction, there are k free generators read as indicated in the hypotheses, then there are k faces of L that have free edges in G and so can be collapsed. Strictly speaking, we should have the subcomplexes K_+ and K_- being 3-manifold complexes in order to apply the free reduction construction, but since the deformations involved in this construction are active only on a regular neighborhood of the 1-skeleton, the conditions above are good enough to make the construction work.

3-manifold like deformations on split 2-complexes. A split 2-complex K has the property that, near the 1-skeleton G, each of its two pieces K_+ and K_- can be treated like a 3-manifold complex provided that the 1-skeleton G is left undisturbed. Also subdivision of the 2-cells by arcs can often be effected as long as the arcs do not meet any projection singularities. Furthermore, it is sometimes possible to switch a 2-cell e_i of K from K_+ to K_- or backwards by level changing deformations that move $e_i \cap N$ from $\mu(J_+)$ to $\mu(J_-)$ or backwards. By passing to the boundaries of the two handlebodies, 3-manifold like deformations of the pieces K_+ and K_- can be explained in terms of isotopies and slidings of simple closed curves. It is these kinds of deformations that are used to prove the deformation theorem, Theorem C, refered to earlier. What is missing in this treatment is a method to control the linked cancellation obstructions under the deformations just mentioned, especially the deformation corresponding to isotopy. For example, suppose we start with a 3-manifold complex K in a manifold $N = M \times [-1, 1]$, and suppose that the relators in the associated presentation $\mathscr{P}_K$ are freely reduced. Can we, by a clever change of basis, achieve the conclusions of Theorem C so that in addition there are no linked cancellation segments at least among the cells that give the free generators as relators? We discuss this further in the next paragraph.

One way to change basis is to change the mapping cylinder structure C_f on J. In turn, this mapping cylinder structure is mostly determined by the curves C_i. We define a change $\{C_i\} \to \{C_i'\}$ to be an AI-*transformation* (algebraic identity transformation) if, up to free reduction, each word $w = w(y_1, \ldots, y_n)$ corresponding to a simple closed curve on BdJ read against the curves C_i is sent to the corresponding word $w(y_1', \ldots, y_n')$ obtained by substituting each y_i' for y_i when read against the curves C_i'. A family of AI-transformations can be constructed rather easily by doing pairs of band transformations where each band transformation is of the type used by Kaneto [16] in giving an alternate proof of Zieschang's result on geometric Whitehead transformations [27]. It can be shown that for any non-singular collection of curves $\{S_i\}$ on the boundary of a handlebody J, and for any collection $\{C_i\}$ of curves associated with a basis for $\pi_1(J)$ via a mapping cylinder structure C_f, there is an AI-transformation that eliminates all linked cancellation segments associated with the curves S_i. Once singularities begin to appear in the curve system $\{S_i\}$, we do not know how to continue: Paired band transformations that eliminate one pair of linked cancellation segments may, because of singularities in the bands, introduce new linked pairs.

Conjecture E. If K is a 2-complex in a 3-manifold M, and if $K \times 0$ 3-deforms in $M \times [-1, 1]$ to a split 2-complex L fixing the 1-skeleton of K, and if the deformations are the 3-manifold like deformations described here, then there is an AI-transformation for a suitable thickening of the 1-skeleton of L so that all associated linked cancellation segments are eliminated.

Remark F. If Conjecture E is true, then knot surgery on a knot in S^3 cannot produce a counter-example to the Poincaré conjecture.

Outline of proof. Let M be a homotopy 3-sphere that results from knot surgery on the 3-sphere S^3. Let K be a 2-spine of M, and let $\mathscr{P}_K = \{y_1, \ldots, y_n \mid r_1, \ldots, r_n\}$ be the associated group presentation. By using the fact that M results from knot surgery, it is not difficult for one to see that adding a trivial relator $*$ to $\mathscr{P}_K$ makes the resulting presentation standard in the sense of extended Nielsen transformations: that is, $\mathscr{P}_K(*) = \{y_1, \ldots y_n | r_1, \ldots, r_n, *\}$ is equivalent under extended Nielsen operations to $\{y_1, \ldots, y_n | y_1, \ldots, y_n, *\}$. But the revised presentation $\mathscr{P}_K(*)$ can be regarded as $\mathscr{P}^2_{K \cup S^2}$ where $K \cup S^2 \subset M$. Let N be a regular neighborhood of $K \cup S^2$ in $M \times [-1, 1]$. Then BdN is the connected sum $2M \cup S^2 \times S^1$. Here $2M$ denotes the double, $M \# -M$, of M. If Conjecture E is true, then $K \cup S^2$ 3-deforms in $M \times [-1, 1]$ to a 2-complex L with no 1-cells and one 2-cell. Then N is the sum of a 4-ball and a 2-handle. But this means that BdN results from S^3 by framed surgery on a knot, and for homological reasons the framing must be the 0-framing. According to Gabai, [14, Cor. 8.3], BdN is irreducible, and this is absurd.

Singular disk systems. Below we describe a general method for producing examples of split 2-complexes. Let M be a 3-manifold and $J \subset M$ a cube with handles. By a *singular disk system* in M we mean a proper map $f: D \to M \setminus IntJ$ where D is a union of disks and where the singularities of f consist of a finite collection of double arcs corresponding to transverse intersections of the image of D in $M \setminus IntJ$. Associated with any singular disk system is a group presentation $\mathscr{P}(J, D, f)$ with generators a set of generators for the free group $\pi_1(J)$ and relators read by the attaching map f for the components of D. Let the double arcs in D

be $A_{1+}, A_{1-}, \ldots, A_{k+}, A_{k-}$. These are proper arcs in D and there are two arcs for each singular component. For each $A_{i\varepsilon}$ ($\varepsilon = +, -$) let $B_{i\varepsilon}$ and $C_{i\varepsilon}$ be two copies of $A_{i\varepsilon}$ parallel to and very near A_i on either side. Now thicken up the arcs $B_{i\varepsilon}$ and $C_{i\varepsilon}$ in M using relative regular neighborhoods to enlarge the handlebody J into one $J(1)$ containing each arc $B_{i\varepsilon}$ and each arc $C_{i\varepsilon}$ in its interior. Let $D(1)$ denote the union of disks $D \cap f^{-1}(M \setminus Int J(1))$. Each component of $D(1)$ contains at most one singular arc $A_{i\varepsilon}$, and $D(1)$ corresponds to a new singular disk system $f|D(1) \to M \setminus Int J(1)$. Now pass to $M \times [-1, 1]$ and convert the map $f|D(1)$ to a non-singular map f_1 by pushing the image of a component of $D(1)$ to the level $1/2$ if it does not contain an arc A_{i-} and pushing the image to the level $-1/2$ if it does contain an arc A_{i-}. Let $G(1)$ be a graph spine of $J(1)$ in the interior of $J(1)$. By using the mapping cylinder structure of $J(1)$ corresponding to a retraction $r: J(1) \to G(1)$, and then using the product structure in $M \times [-1, 1]$, we can complete the collection of disks $f_1(D(1))$ to a 2-complex K with 1-skeleton $G(1)$. It is not difficult to see, using Wright's characterization of formal 3-deformations [25], that the group presentation $\mathscr{P}_K$ associated with K is equivalent under extended Nielsen transformations to the presentation $\mathscr{P}(J, D, f)$. In fact with a little care, we may choose $G(1)$ so that $\mathscr{P}(J, D, f)$ is read in the usual way as the group presentation associated with some cellular subdivision of K. But K is quite clearly a split 2-complex. Up to insertion of cancelling pairs of syllables, any finite group presentation $\mathscr{P}$ can be read as $\mathscr{P}(J, D, f)$ for some suitable singular disk system (see Craggs [11], also Christ [5]); thus we have a large family of examples for our obstruction problem.

The Howie example. By using the attaching curve diagram for the dunce hat spine of the Mazur manifold in Zeeman [26], and adding some trivial half twists to the curve, Howie [15] was able to exhibit a slightly revised spine of the Mazur manifold that comes from a singular disk system, in this case one made up of a single disk. By Theorems B, C, and D, this spine deforms to another split complex 2-spine with an associated presentation that after free reduction has the form, $\{x_1, \ldots, x_n \mid x_1, \ldots, x_n\}$. But since the Mazur manifold is not a ball, there must be linked cancellation segment obstructions, and such obstructions can never be eliminated.

§5. FURTHER REMARKS AND QUESTIONS:

(1) Is there a reasonable theory for cancellation segments? Can they be recognized by some algebraic invariant that exhibits less erratic behaviour under the 3-manifold like operations discussed in §4? One of the things that makes dealing with this obstruction so difficult is that even the cancellation segments cannot be recognized in any functorial way. Recall that they correspond in part to sequences of cancellations, and these are not unique. What is it about non-singular disk systems that allows one to solve the geometric free reduction problem, after Zieschang [27], by passing to a new basis?

(2) There is an algebraic handle cancellation theorem for handle decompositions of simply connected 4-manifolds that shows that all simply connected 4-manifolds have handle decompositions in which all the 1- and 3-handles cancel algebraically. See Craggs [12]. Corresponding to the algebraic cancellation of handles, there are some free reduction problems for pairs of 2-complexes in the 4-manifold which, if resolved, would establish that the 1- and 3-handles could be geometrically cancelled. Can the obstructions developed here be applied to look at the 1- and 3-handle cancellation problem in specific examples?

(3) What is the status of the following restricted Andrews–Curtis conjecture or even the weaker one following it:

Conjecture G. If $\mathscr{P} = \{y_1, \ldots, y_n | r_1, \ldots, r_n\}$ is the group presentation read from a Heegaard diagram of genus n for a homotopy 3-sphere M, then $\mathscr{P}$ is equivalent under extended Nielsen operations to the empty presentation $\{-|-\}$.

Conjecture H. If $\mathscr{P}$ is as above, then $\mathscr{P}(*) = \{y_1, \ldots, y_n | r_1, \ldots, r_n, *\}$ is equivalent under extended Nielsen operations to the standard presentation $\{y_1, \ldots, y_n | y_1, \ldots, y_n, *\}$.

The argument in the outline following Remark F shows that if even the weaker Conjecture H is true, then the Poincaré conjecture reduces to Conjecture E.

Acknowledgements—We want to express our thanks to the members of the Topology Seminars at The University of Illinois and Cornell University who went through an earlier, incorrect version of this paper. Especial thanks are due to Marshal Cohen and Wolfgang Haken for support and encouragement over the course of various improvements to the paper. Roger Livesay and Eric Klassen pointed out to us the linking obstruction in the free reduction construction. Thanks are also due to Jim Howie and the referee for helpful comments. Finally we want to thank Randy Holmes for the high quality drawings that he prepared for us.

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DEFORMATION TABLE

$j =$:	$K(2, k, 2j+1) \nearrow\searrow K(2, k, 2j+2)$ $L(2, k, 2j+1) \rightarrow L(2, k, 2j+2)$	$F(k, 2j+0) \searrow F(k, 2j+1)$	$G(2, k, 2j+1) \nearrow\searrow G(2, k, 2j+2)$
0	trough end inversion	collapse back above $G(k)$	inactive
1	moving down beams (K) moving down intersecting beams (L)	modified side collapse under upper parts of singular rods	inactive
2	same as collapse in Col. 3 (K) # absorbed by deformation of F $j = 3, 4$ (L)	inactive	collapse rods with singularities to lower ends
3	inactive	collapse singular rods down to image of singular point	inactive
4	inactive	collapse triangle under intersecting rods to base	inactive
5	moving down a beam	side collapse under remaining rods other than upper rods	inactive
6	same as collapse in Col. 3 (K) # absorbed by deformation of F $j = 6$ (L)	same as collapse in Col. 3	collapse remaining rods (other than upper rods and lower rods) to lower endpoints
7	moving down a (junction of) beam (s)	side collapse under upper rods	inactive
8	same as collapse in Col. 3 (K) # absorbed by deformation of F $j = 8$(L)	same as collapse in Col. 3	collapse upper rods to lower endpoints
9	inactive	side collapse under upper spokes	inactive
10	inactive	side collapse under dummy spokes	inactive

11	inactive	side collapse under lower spoke	inactive
12	same as collapse in Col. 3 (K) # absorbed by deformation of F $j = 12(L)$	same as collapse in Col. 3	# collapse dummy spokes to hub
13	preliminary cone moves	inactive	inactive
14	desingularizing a cone	inactive	inactive
15	pushing out a cone	inactive	inactive
16	inactive	same as collapse in Col. 3 # absorbed by deformation of L $j = 15$	# collapse upper spokes to hub
17	inactive	same as collapse in Col. 3 # absorbed by deformation of L $j = 15$	# collapse lower spoke
18	moving down a beam	side collapse under lower rod	inactive
19	same as collapse in Col. 3 (K) # absorbed by deformation of F $j = 19(L)$	same as collapse in Col. 3	collapse lower rod to lower endpoint

ABSORPTION TABLE

Col. 1 absorbed by Col. 2	Col. 2 absorbed by Col. 1
$j = 2$ absorbed by $j = 3, 4$	$j = 16$ absorbed by $j = 15$
$j = 6$ absorbed by $j = 6$	$j = 17$ absorbed by $j = 15$
$j = 8$ absorbed by $j = 8$	
$j = 12$ absorbed by $j = 12$	
$j = 19$ absorbed by $j = 19$	

APPENDIX: MAPPING CYLINDER MOVES

Figures 3–8 indicate some fundamental visual deformations in the pullbacks $J^\varepsilon(2)$ and $\Theta_2(J^\varepsilon(2))$. In each case, a complex intersects an object (a ball, a beam segment, or a junction of beam segments) in

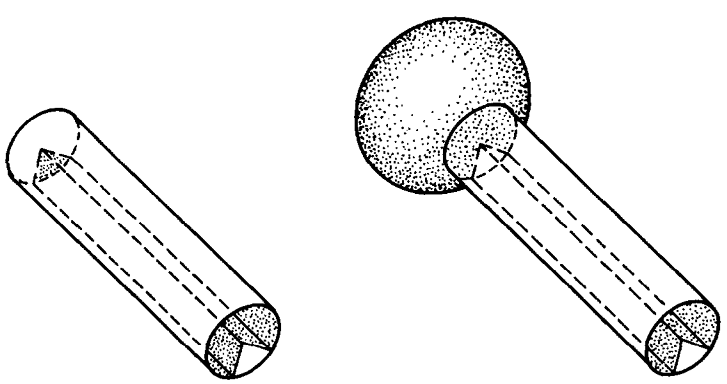

Fig. 3. Inverting a trough end.

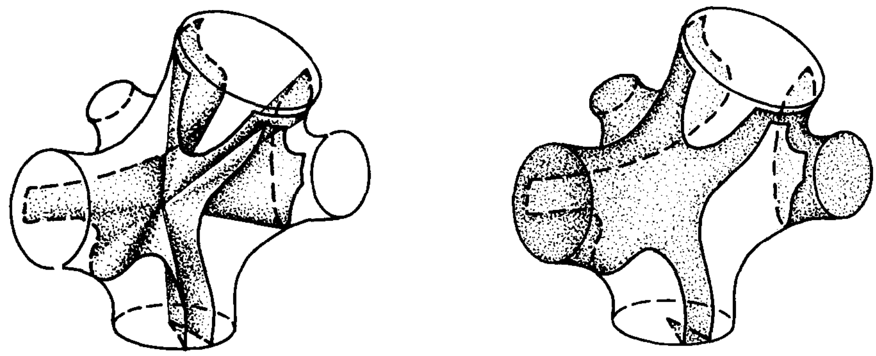

Fig. 4. Pushing out a cone.

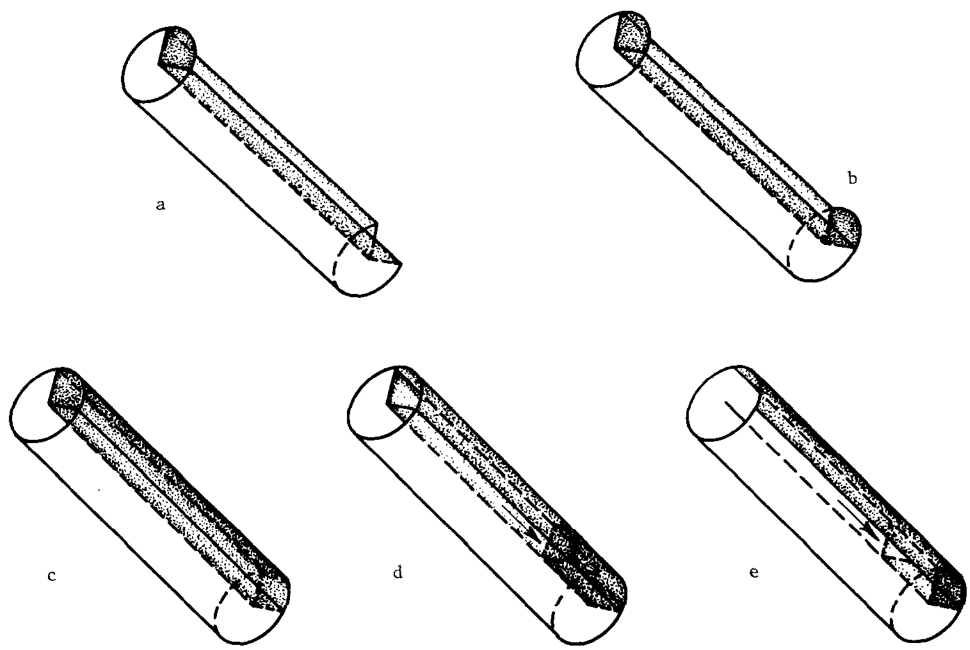

Fig. 5. Moving down a beam.

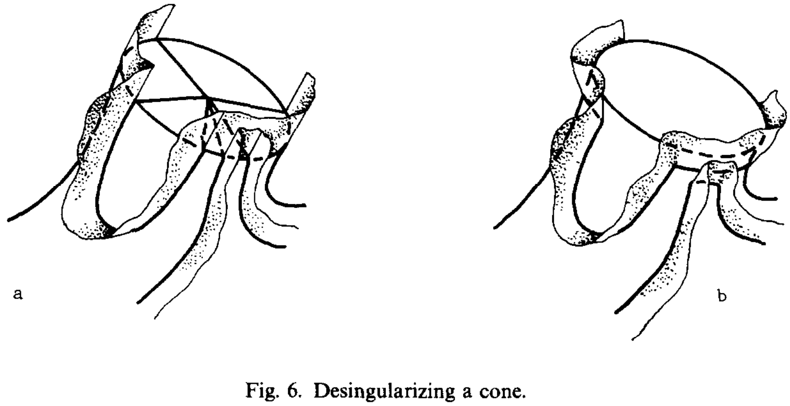

Fig. 6. Desingularizing a cone.

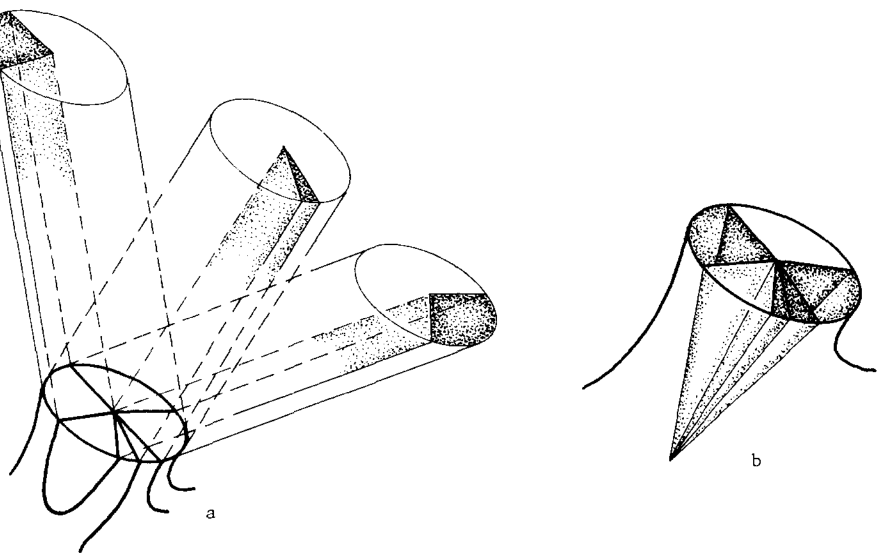

Fig. 7. Moving down a junction of beams.

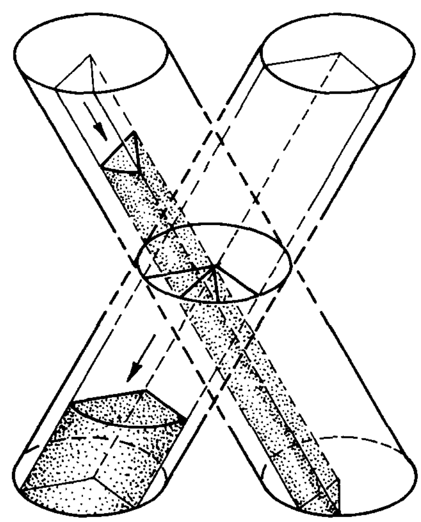

Fig. 8. Moving down intersecting beams.

a certain shaded region, and we describe a deformation of the intersection. The deformations are described in more or less the order in which they come up.

Some pieces of the mapping cylinder beams. Let B be a beam section corresponding to a beam $B^{\varepsilon}(\sigma)$, and let Q be the polyhedron such as is depicted in Fig. 5a or 5b. Then Q will be called a *trough*. Any 2-dimensional intersection of Q with a faces of B will be called a *trough end*. The two rectangular pieces of Q that run down through the interior of B will be called the *trough walls*, and the intersection of the two walls, $B \cap G(2)$, will be called the *trough seam*. The trough cuts the lateral surface of B into two pieces. If one of these pieces intersects, for each trough end present, the trough end in an arc, then we will call that piece the *trough cover*. The only time the definition of trough cover will not make sense is when there are two trough ends, and they are not parallel. We will use the same notation for corresponding images under Θ_2 of the pieces of B.

Some pieces of the mapping cylinder balls. Any time we have a polyhedron Q in a ball section B associated with some ball $B^{\varepsilon}(v)$, we will refer to the polyhedron as a *cone* if it is geometrically a cone from v over a polyhedron in BdB. Sometimes this boundary polyhedron will be a simple closed curve, sometimes a simple closed curve pinched along one or more arcs, and sometimes the sum of one of these curves and a trough end.

Figure 3 (*trough end inversion*). Sometimes it is necessary to change the side of the trough where the upper end is attached. For example, this might be necessary in order to obtain the kind of nesting required for carrying out a moving down a junction of beams move. A small piece of the preceding segment is used to make this move which simply replaces one disk on the boundary of a 3-cell by the complementary disk.

Figure 4 (*Pushing out a cone*). A cone is given from the vertex of a ball over a simple closed curve in the boundary of the ball. This cone is pushed out to the boundary keeping the simple closed curve fixed.

Figure 5 (*Moving down a beam*). The complex meets the beam segment in a trough that is definitely closed off by a disk end on the upper face of the beam segment and may or may not be closed off by a similar disk on the lower face of the beam segment (5a or 5b). It is assumed, in case 5b with two trough ends, that the two trough ends are "parallel" as pictured. If this is not the case then a trough end inversion must be used, as described in Fig. 3, to invert the upper trough end. The trough is expanded into the solid (c). Then the solid is collapsed from the upper end of the trough (d), and the walls of the trough are collapsed leaving just the lower trough end, the trough cover, and the seam (dashed line) where the walls of the trough originally came together.

Figure 6 (*Desingularizing a cone*). A cone is given as in the pushing out a cone move over a now singular simple closed curve. This situation arises because of a previous moving down a junction of beams move. The singularities of the curve come from pinching together parts of the boundary of a surface which lies, except for its boundary, in the exterior of the ball. The exterior surface is peeled back slightly to remove the singularities, and the old cone is simultaneously replaced by the cone over the now non-singular boundary curve.

Figure 7 (*Moving down a junction of beams*). The moves of Fig. 5 are combined for several beams joined along their lower faces. The shadings in the right of Fig. 7b indicate the intersections with the lower face after the expansions $(a) \to (c)$ and $(b) \to (c)$ have been made. An expansion of the type $(b) \to (c)$ can be made only if the lower trough end associated with the expansion is contained in a lower trough end created by a previous expansion of type $(a) \to (c)$. See case (a) of Fig. 9. An expansion of type $(a) \to (c)$ can be made only if the lower trough end to be created intersects all previously created trough ends in at most one point. See case (b) of Fig. 9. At the end of this sequence of moves, we are left with the trough seams, as in Fig. 5, together with trough covers and certain lower trough ends, namely those created by the expansions of type $(a) \to (c)$. These lower trough ends meet pairwise at exactly one point.

Figure 8 (*Moving down intersecting beams*). Here an attempt is made to move down two beams that intersect. The beams are in a 4-manifold, and they intersect in a cross sectional disk common to both. The parts of the 2-complex in the beams, called troughs, intersect exactly in the intersection of the center lines (or trough seams) of the two beams. Just as after the moving down a beam move, only the trough seams, trough covers, and lower trough ends should remain after this move. The trough covers appear to be removed in the figure, but this was done to bring out the hidden view. For each beam the picture ends up as in Fig. 5e. The trough covers should be manipulated so that the covers corresponding to the two beams do not intersect. The singularities that come from intersecting the two revised complexes in the two beams should consist of exactly the intersection of the two beam center lines. To see whether this move is possible, translate the upper trough ends down to the disk of intersection of the two beams. One of three things happens (see Fig. 9): (a) One of the translated trough ends is contained in the other. (b) The two translates intersect in a single point. (c) The interiors of the two translates intersect non-trivially but neither translate is contained in the other. The move can be made in cases (a) and (b). Case (c) represents precisely the linked cancellation segment obstruction. Let the beams be associated with the pair $(\sigma, \sigma') \in \Sigma_0$, say two beams $B(\sigma)$ and $B(\sigma')$, and suppose that

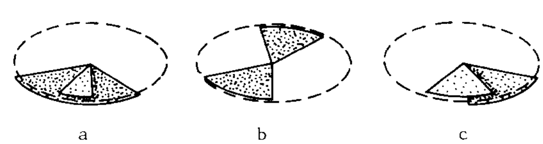

Fig. 9. Nested trough ends.

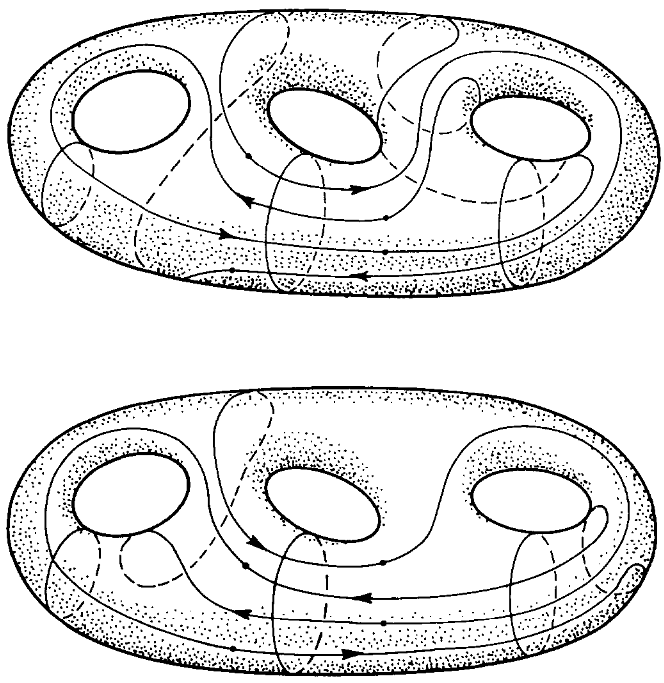

Fig. 10. Linked and unlinked cancellation segments.

activity is scheduled for the inner portions $B^{\varepsilon}(\sigma)$ and $B^{\varepsilon}(\sigma')$ of these two beams (see the first paragraph of Step XII). If case (a) occurs suppose that the translate of the upper trough end associated with σ is contained in the other translate. Use a moving down a beam move first for the beam $B^{\varepsilon}(\sigma)$ then a moving down a beam move for the beam $B^{3/4\varepsilon}(\sigma')$. In order to exhibit the intersecting beams more clearly, the order of the pushes has been made to seem reversed; i.e. it appears that the push is first done down the beam associated with σ', but the reader should note that the push is made down the other beam first. As an alternate procedure, one can use a trough end inversion, Fig. 3, to convert to case (b). In case (b) and two moving down a beam moves can be made in either order and there is no need to adjust the number ε.

Added in proof: Two points were missed in the free reduction construction: (a) There is an additional source of singularities for the map θ_2. These emerge as one passes from the first to the second approximation in Step VI. (b) Some special precautions are necessary in the very last stage of the deformation of L in Step XIII. The two points require minor changes in some steps.

Step III. Add to Σ the set of pairs of distinct edges (σ, σ') such that for some integer k, $\beta(\sigma) = [k, k+1]$ and $\beta(\sigma') = [k-1, k]$ or $\beta(\sigma) = [k-1, k]$ and $\beta(\sigma') = [k, k+1]$, and $\Psi(\sigma) = \Psi(\sigma')$, and some vertices u and u' of σ and σ' with $u \neq u'$ satisfy $\beta(u) = \beta(u') = k$ and $\Psi(u) = \Psi(u')$. Note that this may cause new pairs to be added to Σ_0.

Steps IV–VI: The new pairs added to Σ and Σ_0 account for singularities of the map θ_2 that might remain very near the vertex products $v \times [0,1]$ (v a vertex of G) after passage from the first approximation to the second approximation to θ_2. A new pair in Σ might or might not cause a singularity in θ_2, but just as before, singularities caused by elements of $\Sigma \setminus \Sigma_0$ can be avoided. Condition (5) in Step VI must be changed to reflect the unpredictability of the new singularities: Change "each pair" to "some pairs". The second barycentric subdivision $G''(1)$ of $G(1)$ is a poor choice to have (5) in VI satisfied. We must plan on having $G''(1)$ a derived subdivision of $G(1)$ inducing G'' of G such that Ψ: $G''(1) \to G''$ is simplicial, and the definition of this derived subdivision and hence of $G(2)$ must await a more final form for the map θ_2. Change the order of things in the three steps so that θ_2 is defined by a sequence of three approximations with the derived subdivision being introduced after the second approximation. For the second approximation change (2) to require θ_2 to be linear on each simplex σ, delete (3), and delete the part of (5) requiring $a(\sigma, \sigma') \in \sigma \setminus N(\dot{\sigma}, G(2))$. After the second approximation, go back and define the second derived subdivision $G''(1)$ as a first derived subdivision of the barycentric subdivision of $G(1)$ and define the final third approximation to θ_2 so that (2), (3), and (5) are satisfied.

Step XIII: The very last step of the deformation

$$L(2, n-1, 39) \to L(2, n-1, 40) = L(2, n),$$

may not really be a deformation. There may be some singularities caused by the fact that residual parts of L remain at the level $M \times 0$. To avoid the trouble this causes, observe that (1) $L(2, n-1, 39)$ is locally an open 2-manifold at every point (above level 0) of the lower rod to be disposed of, (2) G is indeed the 1-skeleton of $L(2, n-1, 39)$, and (3) the associated group presentation $\mathscr{P}_{L(2, n-1, 39)}$ already has the desired reduced form. Thus we can stop with $L(2, n-1, 39)$, taking it to be the final $L(2, n)$, and have all the conditions of the theorem satisfied.